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Nonlocal Properties of the Ponderomotive Force in High Intensity Laser Fields

-An Approach Based on the Noncanonical Lie Perturbation Theory-

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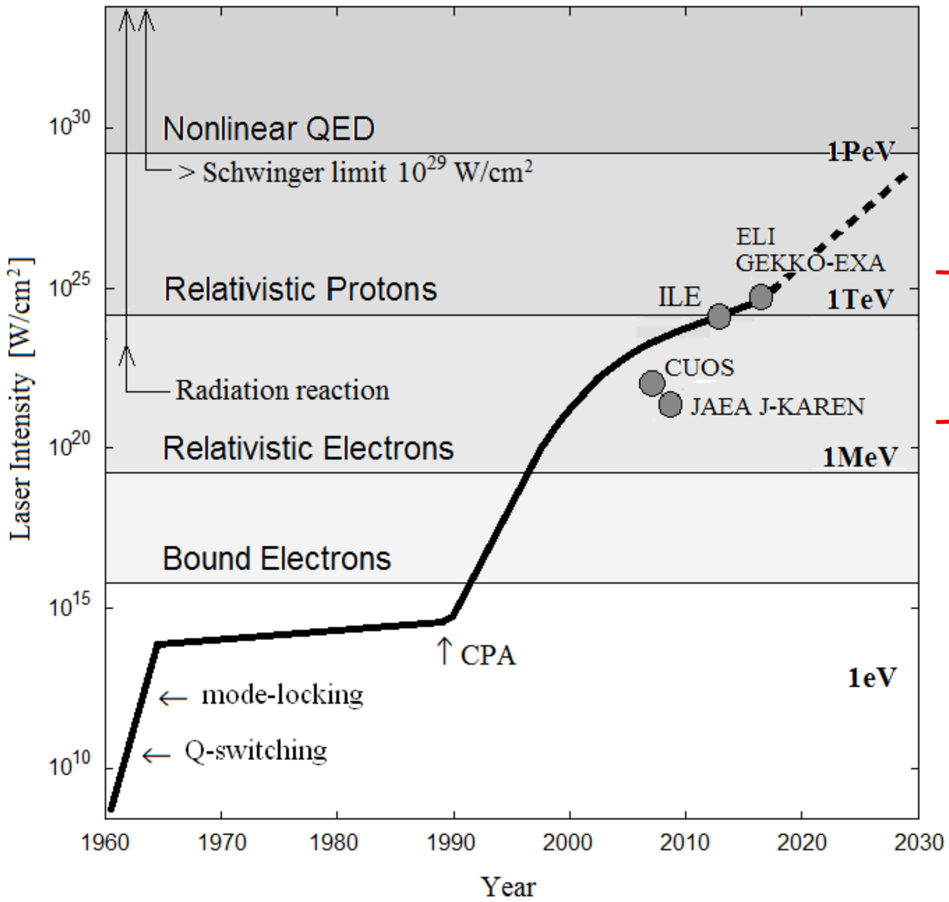
Development of high intensity lasers



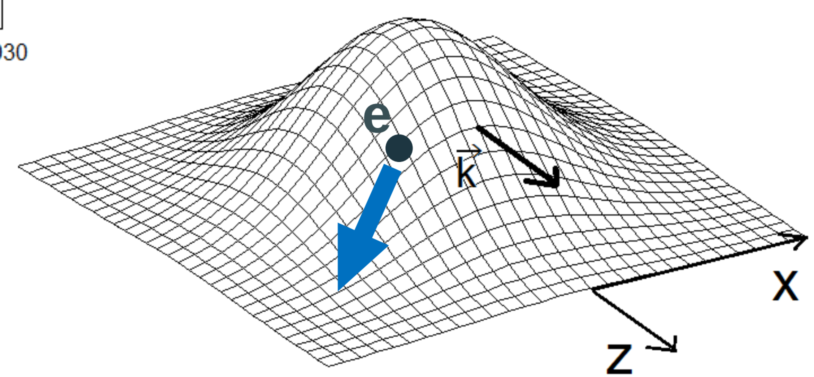
High intensity laser device

GEKKO-ⅩⅡ :   10PW = 10^{16}W, 30μm spot   ⇒ 10^{21} W/cm² (ILE, Osaka Univ.)
JAEA J-KAREN: 0.2PW = 2 × 10^{14}W, 5μm spot   ⇒ 10^{21}W/cm² (KPSI, JAEA)

Strong focusing is necessary.

→ Field becomes non-uniform.

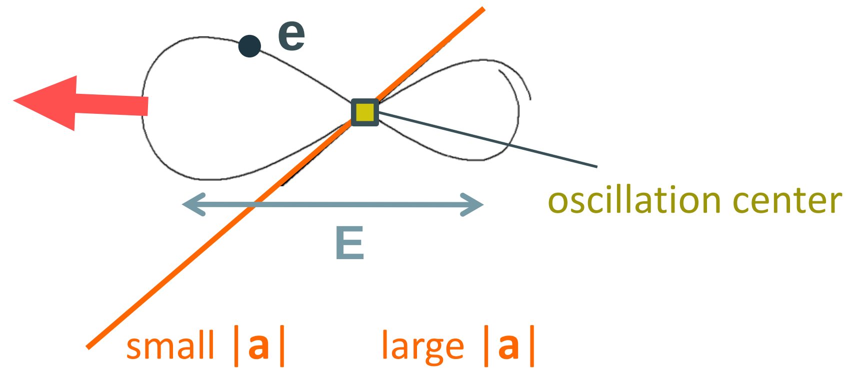
→ Ponderomotive force



Ponderomotive force

$$\boxed{\mathbf{F}_\mathrm{p} = \left\langle \frac{d\mathbf{p}}{d\eta} \right\rangle = -\frac{mc^2}{2\omega}\left\langle \nabla \mathbf{a}^2(x) \right\rangle}$$

Averaged force on
the oscillation center



- Conventional derivation (averaging method)

Equation of motion $\boxed{\frac{d\mathbf{p}}{dt} = mc\left(-\frac{\partial \mathbf{a}}{\partial t} + \frac{\mathbf{p}}{\gamma m} \times (\nabla \times \mathbf{a}) \right)}$

$\mathbf{p} = \mathbf{p}^s + \mathbf{p}^f$ $\quad$ $\mathbf{p}^s = \langle \mathbf{p} \rangle$ one cycle of the laser phase

$$\begin{cases} \text{slow} & \left(\frac{\partial}{\partial \eta} + \mathbf{p}^s \cdot \nabla \right) \mathbf{p}^s_\perp = -\left\langle (\mathbf{p}^f \cdot \nabla) \mathbf{p}^f_\perp \right\rangle \approx \boxed{-\left\langle (\mathbf{a}_\perp(x,\eta) \cdot \nabla) \mathbf{a}_\perp(x,\eta) \right\rangle} \\ \text{fast} & \frac{\partial \mathbf{p}^f_\perp}{\partial \eta} + \cancel{[(\mathbf{p} \cdot \nabla)\mathbf{p}]_{os.}} = -mc \frac{\partial \mathbf{a}_\perp(x,\eta)}{\partial \eta} \end{cases}$$

$\mathbf{p}^f \sim -mc\mathbf{a}_\perp$

[assumption]
gradient of the laser field
amplitude is small

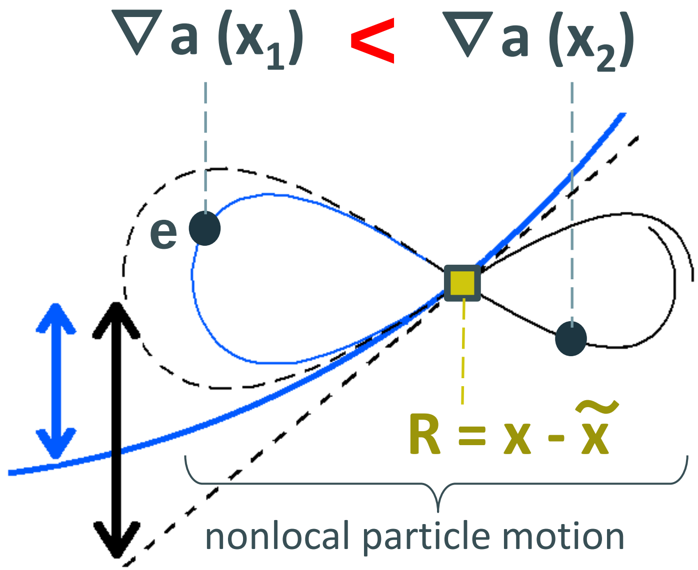
$\eta = \omega t - k z$: phase

Nonlocal effect

$$\left\langle \frac{d\mathbf{p}}{d\eta} \right\rangle = \mathbf{F}_{\mathrm{p}} \propto -\left\langle \nabla \mathbf{a}^2(x) \right\rangle$$

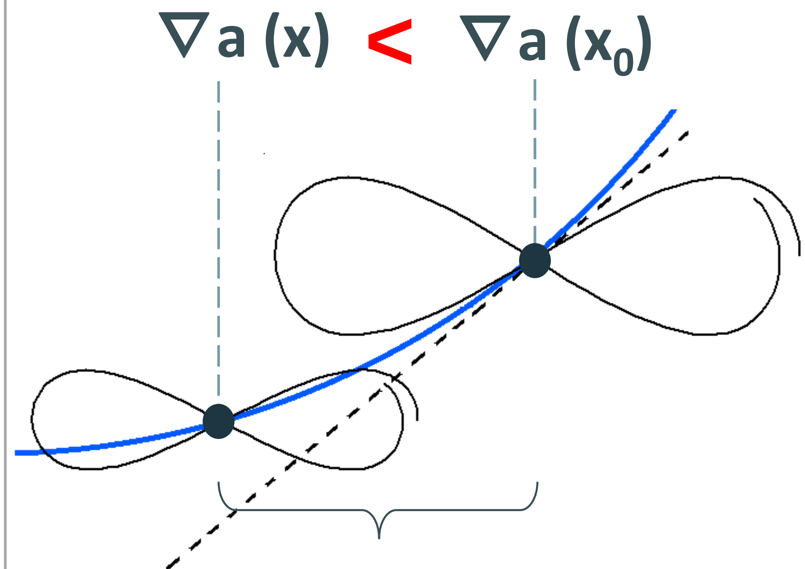
←approximation:
(excursion length) << (scale length of ∇a)

However, under strong focusing, higher-order collections become important.



Perturbation expansion around the oscillation-center

$$a(x) = a(R) + \varepsilon\, \tilde{x}\, \partial_x a(R) + \frac{\varepsilon^2}{2} \tilde{x}^2\, \partial_x^2 a(R) + \cdots$$



around the initial particle position

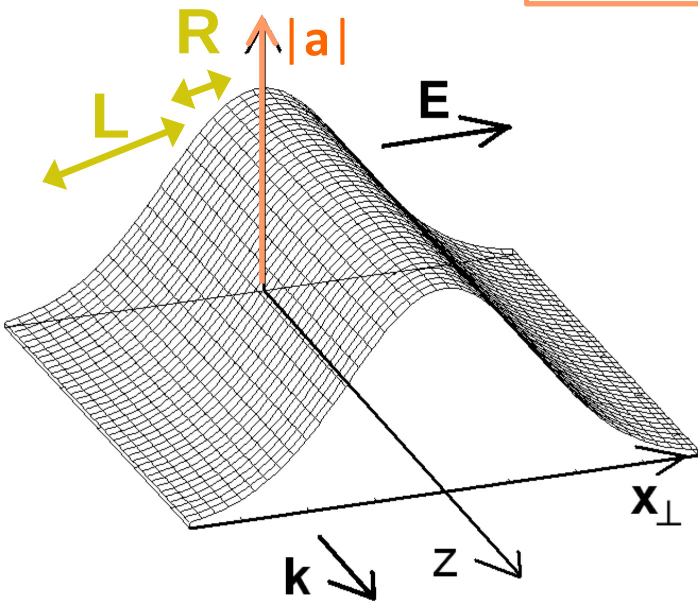
$$a(x) = a(x_0) + \varepsilon (x - x_0) \partial_x a(x_0) + \cdots$$

→ <u>Systematic perturbation analysis based on the Hamiltonian mechanics</u>

Perturbation analysis

Transversely focused laser field

$$\mathbf{a} = a_x(x,y)\sin\eta\ \hat{\mathbf{e}}_x + \varepsilon a_z \cos\eta\ \hat{\mathbf{e}}_z$$



approximately satisfy the Maxwell eqs.: $\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{a}}{\partial t^2} - \nabla^2 \mathbf{a} = 0 \\ \nabla\cdot\mathbf{a} = 0 \end{cases}$

$$\frac{l}{L} \sim O(\varepsilon),\ \frac{l^2}{R} \sim O(\varepsilon^2)$$

l : excursion length in the x-direction

$$L(\mathbf{x}_0) \equiv \left(\frac{\partial_{x_\perp} a(\mathbf{x}_0)}{a(\mathbf{x}_0)}\right)^{-1},\ R(\mathbf{x}_0) \equiv \left(\frac{\partial^2_{x_\perp} a(\mathbf{x}_0)}{a^2(\mathbf{x}_0)}\right)^{-1}$$

: scale length of the amplitude gradient and curvature

Perturbation expansion (around the initial particle position x_0)

$$\begin{aligned} a_x(\mathbf{x}) &= a_{x0} + \varepsilon(\mathbf{x}-\mathbf{x}_0)\cdot\partial_{\mathbf{x}\perp} a_x(\mathbf{x}_0) + \cdots \\ &= a_{x0}\left[1 + \varepsilon\frac{x-x_0}{L} + \varepsilon\frac{y-y_0}{L} + \varepsilon^2\frac{(x-x_0)^2}{2R} + \cdots\right] \end{aligned}$$

Noncanonical Lie perturbation theory

- Perturbation expansion to phase space Lagrangian

$$S = \int L dt = \int \boxed{\gamma_\mu dz^\mu}$$

perturbation

fundamental 1-form

$$\begin{cases} z^\mu \equiv (t; \mathbf{q}, \mathbf{p}) \\ \gamma_\mu \equiv (-h(t, \mathbf{q}, \mathbf{p}); \mathbf{p}, \mathbf{0}) \end{cases}$$

h: Hamiltonian

$\rightarrow \delta S = 0 \rightarrow$ equations of perturbed motion

- Arbitrary **noncanonical variables** are available

  → Make analysis easier

$$h = \sqrt{(mc^2)^2 + c^2(\mathbf{p}_c - mc\mathbf{a})^2}$$

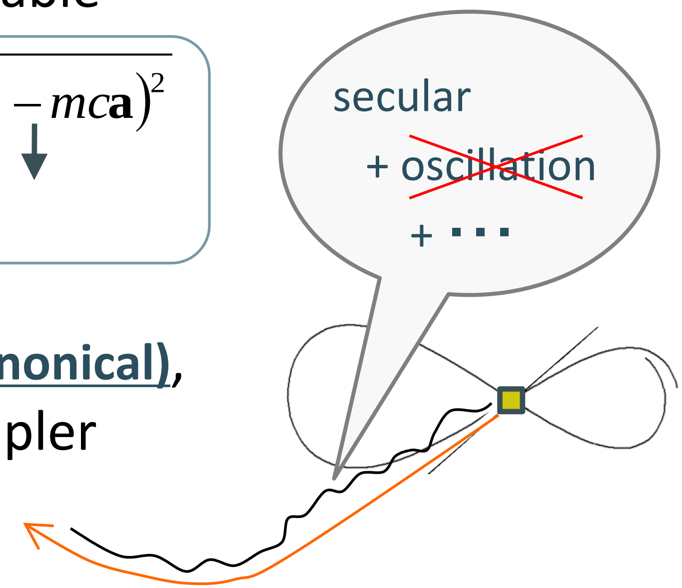
$$\downarrow$$

$$h = \sqrt{(mc^2)^2 + c^2\mathbf{p}^2}$$

- Perturbation analysis

  ← **Lie transformation (near-identity, noncanonical)**,

  Move to a coordinate which gives a simpler expression for the perturbed motion.



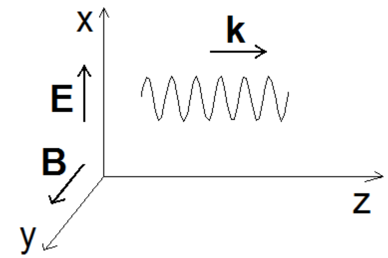
Preparatory transformation

canonical coordinate

$$\begin{cases} z^{\mu} = \left(t;\, x,\, y,\, z,\, p_{cx},\, p_{cy},\, p_{cz}\right) \\ \gamma_{\mu} = \left(-h\,;\, p_{cx},\, p_{cy},\, p_{cz},\, 0, 0, 0\right), \end{cases}$$

relativistic Hamiltonian:

$$h = \sqrt{\left(mc^2\right)^2 + c^2\left(\mathbf{p}_c - mc\mathbf{a}(t, x, y, z)\right)^2}$$



$$\mathbf{a} = \left(a_x(\mathbf{x}_\perp, \eta),\, 0,\, \varepsilon a_z(\mathbf{x}_\perp, \eta)\right)$$

$$\gamma_\mu dz^\mu = \Gamma_\mu dZ^\mu$$

noncanonical transformation

- **mechanical momentum** → Hamiltonian w/o explicit **a**-dependence
- **independent variable** $\eta = \omega t - k\,z$ (phase)
- **invariant of the motion** $p_\eta = p_z - \gamma mc$

simplify

Poisson tensor

$$\mathbf{J}^{(0)} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$$

simplify

noncanonical coordinate

$$\begin{cases} z^{\mu} = \left(\eta;\, x,\, y,\, z,\, p_x,\, p_y,\, p_\eta\right) \\ \gamma_{\mu} = \left(-K\,;\, p_x + mca_x(\mathbf{x}_\perp, \eta),\;\; p_y,\;\; p_\eta + \varepsilon mca_z(\mathbf{x}_\perp, \eta), 0, 0, 0\right), \end{cases}$$

new Hamiltonian:

$$\boxed{K = -\frac{1}{2kp_\eta}\left(m^2c^2 + \mathbf{p}_\perp^2 + p_\eta^2\right)}$$

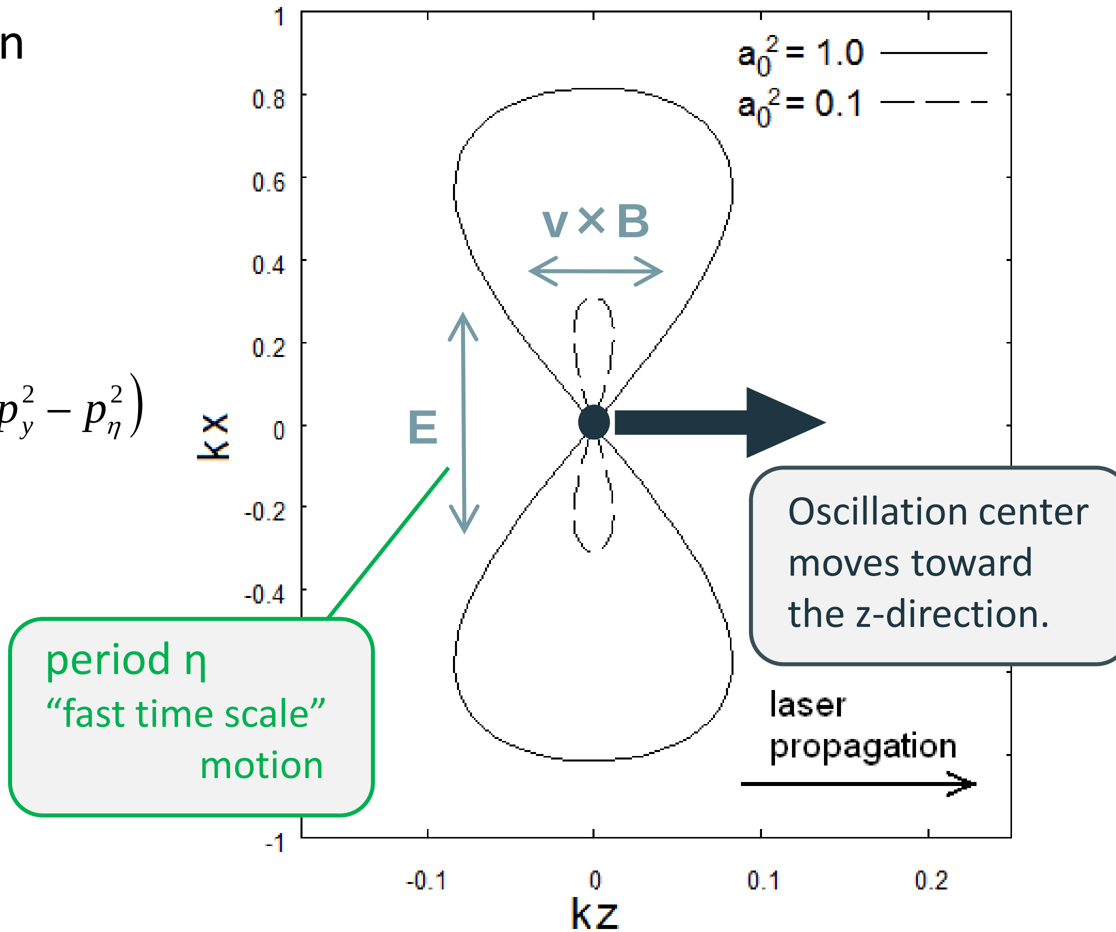
Particle trajectory in a uniform laser field

$O(\varepsilon^0)$

$$\begin{cases} z^{(0)\mu} = \left(\eta; x, y, z, p_x, p_y, p_\eta\right) \\ \gamma_\mu^{(0)} = \left(-K\,;\, p_x + mca_{x0}\sin\eta,\ \ p_y,\ \ p_\eta\,, 0, 0, 0\right),\ K = -\dfrac{1}{2kp_\eta}\left(m^2c^2 + \mathbf{p}_\perp^2 + p_\eta^2\right) \end{cases}$$

→ 0th-order equations of motion

$$\begin{cases} \dfrac{dx}{d\eta} = -\dfrac{\partial\gamma_0^{(0)}}{\partial p_x} = -\dfrac{p_x}{k_z p_\eta} \\ \dfrac{dy}{d\eta} = -\dfrac{\partial\gamma_0^{(0)}}{\partial p_y} = -\dfrac{p_y}{k_z p_\eta} \\ \dfrac{dz}{d\eta} = -\dfrac{\partial\gamma_0^{(0)}}{\partial p_\eta} = \dfrac{1}{2k_z p_\eta^2}\left(m^2c^2 + p_x^2 + p_y^2 - p_\eta^2\right) \\ \dfrac{dp_x}{d\eta} = -\dfrac{\partial\gamma_1^{(0)}}{\partial\eta} = -mca_{x0}\cos\eta \\ \dfrac{dp_y}{d\eta} = -\dfrac{\partial\gamma_2^{(0)}}{\partial\eta} = 0 \\ \dfrac{dp_\eta}{d\eta} = -\dfrac{\partial\gamma_3^{(0)}}{\partial\eta} = 0 \end{cases}$$

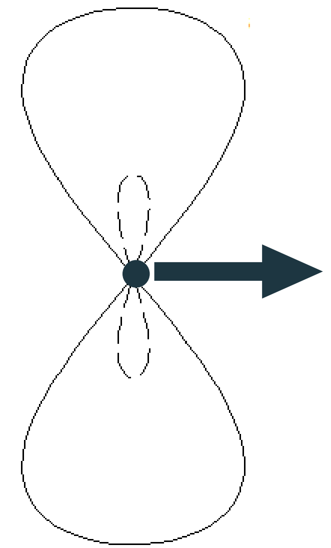


Transformation to oscillation-center

noncanonical coordinate

$$\begin{cases} z^\mu = \left(\eta;\, x,\, y,\, z,\, p_x,\, p_y,\, p_\eta\right) \\ \gamma_\mu = \left(-K\,;\, p_x + mca_x(\mathbf{x}_\perp,\eta),\;\; p_y,\;\; p_\eta + \varepsilon mca_z(\mathbf{x}_\perp,\eta), 0, 0, 0\right) \end{cases}$$

$$K = -\frac{1}{2kp_\eta}\left(m^2c^2 + \mathbf{p}_\perp^2 + p_\eta^2\right)$$: Hamiltonian



noncanonical transformation

$$\gamma_\mu dz^\mu = \Gamma_\mu dZ^\mu$$

$Z^i = z^i - \tilde{z}^{i(0)};\, i = 1,\cdots 5$

$\tilde{z}^{i(0)}$: oscillatory components of the figure-eight motion

oscillation-center coordinate

$$\begin{cases} Z^\mu = \left(\eta;\, X, Y, Z, P_x, P_y, p_\eta\right) \\ \Gamma_\mu = \left(-\underline{\underline{\kappa}};\, P_x + \tilde{p}_x^{(0)} + mc\underline{\underline{a_x(\mathbf{X}_\perp,\eta)}}, P_y,\, p_\eta + \varepsilon mc\underline{\underline{a_z(\mathbf{X}_\perp,\eta)}}, 0, 0, 0\right) \end{cases}$$

κ : new Hamiltonian

Perturbation: $a_x(\mathbf{X}_\perp,\eta) = a_{x0} + \varepsilon\left(\mathbf{X} + \tilde{x}^{(0)} - \mathbf{x}_0\right)\cdot\partial_{\mathbf{x}\perp}a_x(\mathbf{x}_0) + \dfrac{\varepsilon^2}{2}\left(X + \tilde{x}^{(0)} - x_0\right)^2\partial_x^2 a_x(\mathbf{x}_0) + \cdots$

Secular particle motion

$O(\varepsilon^0)$

$$\begin{cases} \dfrac{d\mathbf{P}_\perp^{(0)}}{d\eta} = -\dfrac{\partial\kappa}{\partial \mathbf{X}_\perp} = 0 \\ \dfrac{dp_\eta^{(0)}}{d\eta} = -\dfrac{\partial \kappa}{\partial Z} = 0 \end{cases}$$

--- Invariant

$$\Rightarrow \mathbf{P}^{(0)} = \left(0,\, 0,\, \frac{mc}{2\zeta_0}\left(\frac{a_{x0}^2}{2} + 1 - \zeta_0^2\right)\right)$$

Oscillation center moves only to the z-direction due to the laser momentum.

■ Initial condition

$$\mathbf{p}_0 = (0, 0, p_{z0}) \quad \text{at } \eta = 0$$

$$p_{\eta 0} = p_{z0} - \gamma_0 mc \equiv -mc\zeta_0$$

$$\zeta_0 = 1 \quad \text{for } p_{z0} = 0$$

$O(\varepsilon^1)$

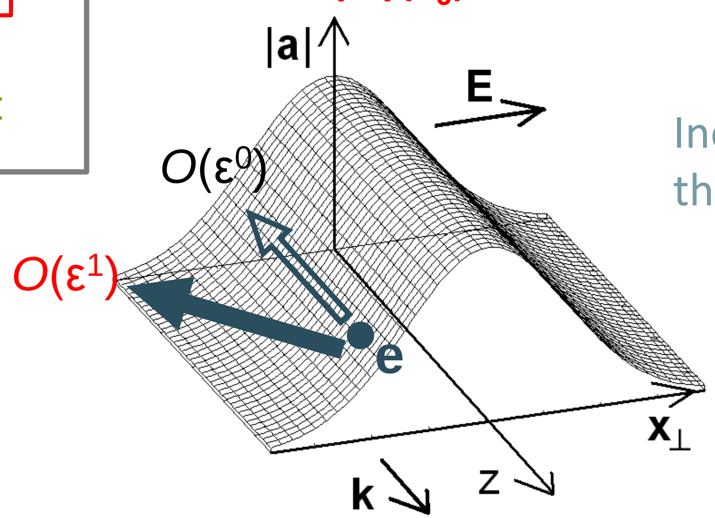
Lie transformed coordinate

$$Z'^\mu = (\eta;\, X', Y', Z', P'_x, P'_y, p'_\eta), \quad \boxed{\overline{Z'^\mu} = \overline{Z}^\mu}$$

$$\begin{cases} \dfrac{d\mathbf{P}'_\perp}{d\eta} = \boxed{\varepsilon \dfrac{mca_{x0}^2}{2k_z L(x_0)} \dfrac{mc}{p'_\eta}}\, \hat{\mathbf{e}}_\perp \\ \dfrac{dp'_\eta}{d\eta} = 0 \end{cases}$$

--- Invariant

Ponderomotive force ∝ ∇|a|(x_0)



All oscillatory components can be obtained by the backward transformation:

$$z = \frac{a_{x0}}{4} l \left[\boxed{\left(1 - \varepsilon\frac{2l}{L}\right)\left(\eta - \frac{1}{2}\sin 2\eta\right)} \right.$$
$$\left. \varepsilon \frac{2l}{L}\left(\sin\eta - \eta\cos\eta + \frac{1}{4}(\sin\eta - \sin 3\eta)\right)\right]$$

Increase/decrease of the 0th-order motion

$$(\zeta_0 = 1)$$

2nd-order modulation to the ponderomotive force

Equations of motion (x-direction, up to the 2nd order)

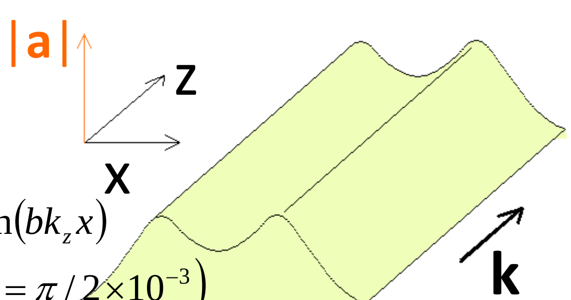
$$\begin{cases} \dfrac{dX''}{d\eta} = -\dfrac{P''_x}{k_z p''_\eta} \\[2ex] \dfrac{dP''_x}{d\eta} = -\dfrac{mca_x(x_0)}{2} l \left[\boxed{\dfrac{1}{L(x_0)}} + \underline{\left(\dfrac{1}{L^2(x_0)} + \dfrac{1}{R(x_0)} \right)(X'' - x_0)} \right] \end{cases}$$

ponderomotive force evaluated at the initial particle position **2nd-order modulation**

$$\Rightarrow \boxed{\frac{d^2 P''_x}{d\eta^2} = -\frac{l^2}{2}\left(\frac{1}{L^2} + \frac{1}{R}\right) P''_x}$$

* $R > 0 \Rightarrow$ betatron-like oscillation

  period: $\theta = l\sqrt{\dfrac{1}{2}\left|\dfrac{1}{L^2} + \dfrac{1}{R}\right|}$



$|\mathbf{a}| = a_0 \cosh(bk_z x)$

$(a_0 = 10,\ b = \pi / 2\times10^{-3})$

$\Rightarrow$ particle confinement

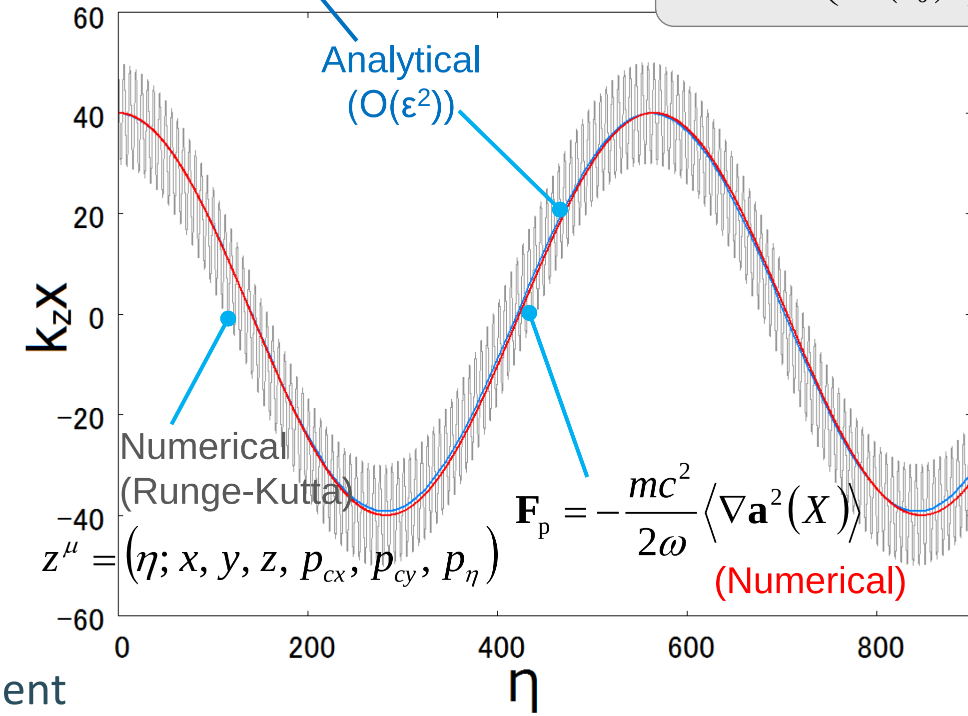
* vector potential

$$\boxed{\mathbf{a} = a_x(x)\sin\eta\ \hat{\mathbf{e}}_x}$$

for simplicity.

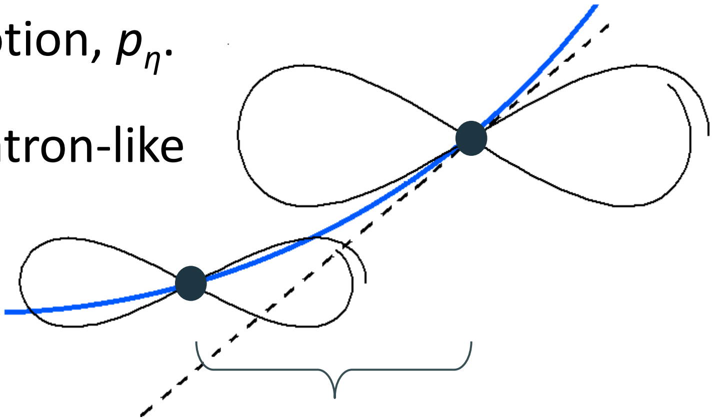
$$L(x_0) \equiv \left(\frac{\partial_x a(x_0)}{a(x_0)}\right)^{-1}$$

$$R(x_0) \equiv \left(\frac{\partial_x^2 a(x_0)}{a(x_0)}\right)^{-1}$$



Summary

- Noncanonical Lie perturbation method was applied to systematically analyze particle motion affected by the relativistic ponderomotive force.
- We found a coordinate suitable for the analysis including phase η and invariant of the unperturbed motion, p_η.
- Analytical solution corresponding to the betatron-like oscillation was found which suggests possibility of particle confinement in an interaction region with laser fields.



Future work

- Noncanonical Lie perturbation analysis around the oscillation-center
- Investigate higher-order (3rd order ~) collections to the ponderomotive force